Occasional Note Met O 21 76/1

METEOROLOGICAL OFFICE
23 JUN1976
LIBRARY

122229

NUMERICAL INTEGRATION OF FLOW IN A ROTATING ANNULUS II: THREE DIMENSIONAL MODEL

L Farnell

and

R A Plumb

Geophysical Fluid Dynamics Laboratory

Meteorological Office

Bracknell, Berkshire

February 1976

Contents

	Definition of symbols	(ii)
1	Introduction	1
2	Equations governing the system	2
	2.1 The governing equations	2
	2.2 Integrals of the governing equations	3
3	The grid scheme and initialisation of data	7
4	The finite difference equations	9
	4.1 Finite-difference form of the governing equations	9
	4.2 Integrals of the finite difference equations	11
	4.3 The Poisson equation	18
5	Method of solution	20
	5.1 The Poisson equation	20
	5.2 Outline of procedure	22
6	Summary	24
	References	24

Definition of symbols

a	inner radius of annulus
A	(total angular momentum)/$\eta\rho_0$
$\mathcal{A}$	advection operator
b	outer radius of annulus
c	specific heat of fluid
C_{PK}	energy conversion rate
d	depth of annulus
$\mathcal{D}$	diffusion operators
$\mathcal{E}_K$	(total kinetic energy)/$\eta\rho_0$
$\mathcal{E}_P$	(total potential energy)/$\eta\rho_0$
$\mathcal{F}$	flux terms
g	acceleration due to gravity
i	radial grid index
I	number of principal grid points in radial direction
j	azimuthal grid index
J	number of principal grid points in azimuthal direction
k	vertical grid index
K	number of principal grid points in vertical direction
m	time index
$\mathcal{M}$	mass flux contribution to angular momentum balance
η	fraction of annulus integrated = $1/\eta$
Q	(total heat content)/$\eta c\rho_0$
r	radial coordinate
r_i	co-ordinate of principal radial grid points
$\tilde{r}_i$	co-ordinate of secondary radial grid points
t	time
T	temperature
T^m_{ijk}	temperature at $r=\tilde{r}_i, \theta=\theta_j, z=\tilde{z}_k, t=m.\Delta t$
T_0	reference temperature

(u, v, w)	velocity components in (r, θ, z)
u^m_{ijk}	radial velocity at $r = r_i, z = \tilde{z}_k, \theta = \tilde{\theta}_j, t = m.\Delta t$
v^m_{ijk}	azimuthal velocity at $r = \tilde{r}_i, z = z_k, \theta = \theta_j, t = m.\Delta t$
w^m_{ijk}	vertical velocity at $r = \tilde{r}_i, z = z_k, \theta = \tilde{\theta}_j, t = m.\Delta t$
z	vertical co-ordinate
z_k	co-ordinate of principal vertical grid points
$\tilde{z}_k$	co-ordinate of secondary vertical grid points
α	volume coefficient of expansion
δ	finite difference derivative
$\delta r_i, \delta \tilde{r}_i$	radial grid increments
$\delta z_k, \delta \tilde{z}_k$	vertical grid increments
$\delta\theta$	azimuthal grid increment $(= \Phi / J)$
Δ	divergence
Δt	timestep
θ	azimuthal co-ordinate
θ_j	co-ordinate of principal azimuthal grid points
$\tilde{\theta}_j$	co-ordinate of secondary azimuthal grid points
κ	coefficient of thermal diffusion
ν	coefficient of kinematic viscosity
Π	pressure / ρ_0
ρ	density perturbation
ρ_0	mean density
ρ_1, ρ_2	expansion coefficients
σ	buoyancy
Φ	angle spanned by azimuthal grid points $(= 2\pi / n)$

1 INTRODUCTION

This report describes the formulation of a numerical model to integrate the Navier-Stokes equations for a liquid contained in a rotating annulus. This model is essentially an extension to three dimensions of that previously described (Farnell and Plumb, 1975, hereafter referred to as FP) for integration of the equations for axisymmetric motions; basic parameters of the system such as its boundaries and the form of variation allowed for viscosity and conductivity are unchanged. As with the axisymmetric model the program is written to allow a variety of boundary conditions to be used.

The present model is similar to that of Williams (1969) in many respects. The major differences are as follows.

(i) Non-uniform grid

As with the axisymmetric model, the grids in the radial and vertical directions are stretched so as to resolve the boundary layers. However a regular grid is retained in the azimuthal direction.

(ii) Diffusion terms

The Dufort-Frankel scheme is used to represent the diffusion terms (see FP § 4.2). Furthermore, since the viscosity is allowed to vary, several (small) terms involving the gradient of viscosity, absent in Williams' equations, are included.

(iii) Angular momentum advection

The non-linear terms in the zonal equation of motion are formulated as in the axisymmetric model (FP § 4.1).

(iv) Solution of the Poisson equation

The use of a uniform azimuthal grid allows the Poisson equation for pressure to be solved using Fourier transforms in the azimuthal direction, which automatically applies the boundary condition of periodicity. However an iterative technique must be used for the remaining directions because of the non-uniform grids, which do not allow Williams' second eigenfunction expansion.

Because of the amount of data required to represent a field in three dimensions, the grid resolution is inevitably restricted by the capacity of the computer. In this model the number of grid points in any meridional section (ie

at constant azimuth) is up to 256, which in general means 16 radial and 16 vertical points. The number of meridional sections is constrained by the requirements of the fast Fourier Transform to be of the form 2^n, which in practice means 16, 32 or 64. To allow good resolution of wave motions it is possible to examine just one period of the wave, though this necessarily means that only harmonics of the basic frequency may be invoked. In other words, the azimuth range is $(0, \Phi)$, where $\Phi = \frac{2\pi}{n}$ and n is an integer.

The continuous equations and their integrals are presented in Section 2. In section 3, the grid point scheme and the way in which the velocity and temperature fields are initialised is described. The finite-difference equations are given in Section 4, together with the finite difference form of the physically important integrals. An outline of the method by which the solution is evaluated is given in Section 5.

2 EQUATIONS GOVERNING THE SYSTEM

For ease of comparison, these equations will be numbered as in the description of the axisymmetric system (FP 2.1-2.29).

2.1 The Governing Equations

(i) The equations of motion are modified only in the advective and diffusion terms plus inclusion of the pressure gradient in the zonal equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial \pi}{\partial r} + v\left(2\Omega + \frac{v}{r}\right) \ldots\ldots$$

$$\ldots\ldots + \Delta^2(r,u) + \frac{1}{r}\frac{\partial (ru)}{\partial r}\frac{\partial r}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} - \frac{2}{r^2}\frac{\partial (rv)}{\partial \theta} + \frac{1}{r^2}\frac{\partial (rv)}{\partial r}\frac{\partial r}{\partial \theta} \qquad (2.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial v}{\partial \theta} = -\frac{1}{r}\frac{\partial \pi}{\partial \theta} - u\left(\frac{v}{r} + 2\Omega\right) \ldots\ldots$$

$$\ldots\ldots + \Delta^2(r,v) + \frac{1}{r}\frac{\partial u}{\partial \theta}\frac{\partial r}{\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta}\frac{\partial r}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial r}{\partial z} + \frac{2}{r^2}\frac{\partial (ru)}{\partial \theta} \qquad (2.2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} + \frac{v}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial \pi}{\partial z} - \frac{g\rho}{\rho_0} \quad \ldots\ldots \qquad (2.3)$$

$$\ldots\ldots + \tilde{\Delta}^2(\nu, w) + \frac{\partial w}{\partial z}\frac{\partial \nu}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial \nu}{\partial z} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial \nu}{\partial \theta}$$

(ii) The continuity equation

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0 \qquad (2.4)$$

(iii) The heat equation is modified in a similar way to the equations of motion

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} + \frac{v}{r}\frac{\partial T}{\partial \theta} = \tilde{\Delta}^2(\kappa, T) \qquad (2.5)$$

The diffusion operators are modified as follows:

$$\Delta^2(\lambda, A) = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{\lambda r^3 \frac{\partial}{\partial r}\left(\frac{A}{r}\right)\right\} + \frac{\partial}{\partial z}\left(\lambda \frac{\partial A}{\partial z}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial A}{\partial \theta}\right) \qquad (2.6)$$

$$\tilde{\Delta}^2(\lambda, A) = \frac{1}{r}\frac{\partial}{\partial r}\left\{\lambda r \frac{\partial A}{\partial r}\right\} + \frac{\partial}{\partial z}\left(\lambda \frac{\partial A}{\partial z}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial A}{\partial \theta}\right)$$

The equation of state (FP 2.7) remains unaltered.

$$\frac{\rho}{\rho_0} = \rho_1 (T - T_0)^2 + \rho_2 (T - T_0)^2 \qquad (2.7)$$

2.2 Integrals of the Governing Equation

These need little modification, except that the integral over azimuth, which leads to the factor 2π in all the definitions in FP § 2.2, cannot now be separated.

(a) Angular Momentum Balance

The total angular momentum of the fluid is $\eta\rho_0 A$ where η is the wave-

number $2\pi/\Phi$, and

$$A = \int_0^d \int_0^{\Phi} \int_a^b v r^2 \, dr \, d\theta \, dz \tag{2.8}$$

In the equation for the evolution of angular momentum,

$$\frac{\partial A}{\partial t} = \mathcal{J}_A^{(D)} + \mathcal{J}_A^{(R)} + \mathcal{J}_A^{(V)} \tag{2.9}$$

the dynamical flux through the boundaries

$$\mathcal{J}_A^{(D)} = -\int_0^{\Phi} \int_a^b [wv]_{z=0}^{z=d} r^2 \, dr \, d\theta - \int_0^{\Phi} \int_0^d [r^2 uv]_{r=a}^{r=b} \, dz \, d\theta, \tag{2.10}$$

and the contribution due to advection of the rotational angular momentum of the annulus

$$\mathcal{J}_A^{(R)} = -2\Omega \int_a^b \mathcal{M}(r) . r \, dr \tag{2.11}$$

are essentially unchanged. The mass flow across a surface is given by

$$\mathcal{M}(r) = \int_0^d \int_0^{\Phi} r u \, d\theta \, dz \tag{2.12}$$

which becomes

$$\mathcal{M} = a \int_0^d \int_0^{\Phi} u(a,z) \, d\theta \, dz = b \int_0^d \int_0^{\Phi} u(b,z) \, d\theta \, dz \tag{2.14}$$

in the special case of rigid upper and lower boundaries.

The viscous flux of angular momentum is now defined as follows

$$\mathcal{J}_A^{(v)} = \int_0^d \int_0^{\Phi} \left[\nu r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]_{r=a}^{r=b} d\theta\, dz + \int_0^{\Phi} \int_a^b \left[\nu \frac{\partial v}{\partial z}\right]_{z=0}^{z=d} r^2\, dr\, d\theta \ldots.$$

$$\ldots. + \int_0^d \int_0^{\Phi} \int_a^b \left\{ r \frac{\partial u}{\partial \theta}\frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta}\frac{\partial v}{\partial \theta} + r \frac{\partial w}{\partial \theta}\frac{\partial v}{\partial z} \right\} dr\, d\theta\, dz \qquad (2.15)$$

(b) Heat Balance

The heat content of the fluid is $n\rho_0 c Q$, where

$$Q = \int_0^d \int_0^{\Phi} \int_a^b T r\, dr\, d\theta\, dz \qquad (2.16)$$

The heat flux terms in

$$\frac{dQ}{dt} = \mathcal{J}_Q^{(D)} + \mathcal{J}_Q^{(V)} \qquad (2.17)$$

are essentially unchanged since the extra terms introduced by the full three dimensional equations are identically zero. The dynamical heat flux is

$$\mathcal{J}_Q^{(D)} = -\int_0^d \int_0^{\Phi} \left[ruT\right]_{r=a}^{r=b} d\theta\, dz - \int_0^{\Phi} \int_a^b \left[wT\right]_{z=0}^{z=d} r\, dr\, d\theta \qquad (2.18)$$

and the diffusive flux of heat from the boundaries is

$$\mathcal{J}_Q^{(v)} = \int_0^d \int_0^{\Phi} \left[\kappa r \frac{\partial T}{\partial r}\right]_{r=a}^{r=b} d\theta\, dz + \int_0^{\Phi} \int_a^b \left[\kappa \frac{\partial T}{\partial z}\right]_{z=0}^{z=d} r\, dr\, d\theta \qquad (2.19)$$

(c) Kinetic Energy

The total kinetic energy is $n\rho_0 \mathcal{E}_K$, where

$$\mathcal{E}_K = \frac{1}{2} \int_0^d \int_0^\Phi \int_a^b (u^2+v^2+w^2)\, r\, dr\, d\theta\, dz. \tag{2.20}$$

In the equation for the evolution of kinetic energy

$$\frac{d\mathcal{E}_K}{dt} = C_{PK} + \mathcal{J}_K^{(0)} - D_K, \tag{2.21}$$

the rate of conversion of potential to kinetic energy

$$C_{PK} = -\int_0^d \int_0^\Phi \int_a^b g w \frac{\rho}{\rho_0}\, r\, dr\, d\theta\, dz \tag{2.22}$$

and the flux of kinetic and internal energy into the fluid at the boundaries

$$\mathcal{J}_K^{(0)} = -\int_0^d \int_0^\Phi \left[ru\left\{\pi + \frac{1}{2}(u^2+v^2+w^2)\right\}\right]_{r=a}^{r=b} d\theta\, dz \ldots\ldots$$

$$\ldots\ldots - \int_0^\Phi \int_a^b \left[w\left\{\pi + \frac{1}{2}(u^2+v^2+w^2)\right\}\right]_{z=0}^{z=d} r\, dr\, d\theta \tag{2.23}$$

are essentially unaltered.

The rate of dissipation of kinetic energy is modified both by the modification of the diffusion operators (2.6) and by the inclusion of extra terms:

$$D_K = -\int_0^d \int_0^\Phi \int_a^b \left\{ u\left[\Delta^2(\nu,u) + \frac{1}{r}\frac{\partial(ru)}{\partial r}\frac{\partial \nu}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial \nu}{\partial z} + \frac{1}{r^2}\frac{\partial(\nu r)}{\partial \theta}\frac{\partial \nu}{\partial \theta} \ldots\ldots\right.\right.$$

$$\ldots\ldots - \left.\frac{2}{r^2}\frac{\partial(\nu v)}{\partial \theta}\right] + w\left[\tilde{\Delta}^2(\nu,w) + \frac{\partial u}{\partial z}\frac{\partial \nu}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial \nu}{\partial z} + \frac{1}{r}\frac{\partial \nu}{\partial z}\frac{\partial \nu}{\partial \theta}\right] \ldots \tag{2.24}$$

$$\left.\left. \ldots + v\left[\Delta^2(\nu,v) + \frac{2}{r^2}\frac{\partial(u\nu)}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\frac{\partial \nu}{\partial r} + \frac{1}{r^2}\frac{\partial \nu}{\partial \theta}\frac{\partial \nu}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial \nu}{\partial z}\right]\right\} r\, dr\, d\theta\, dz$$

(d) Potential Energy

This is essentially unchanged apart from the modification of the diffusion operator in (2.29). Defining

$$\mathcal{E}_p = \int_0^d \int_c^{\Phi} \int_a^b g \frac{\rho}{\rho_0} r z \, dr \, d\theta \, dz \tag{2.25}$$

so that $n\rho_0\mathcal{E}_p$ is the total potential energy of the system, we have

$$\frac{d\mathcal{E}_p}{dt} = \frac{g}{\rho_0} \int_0^d \int_0^{\Phi} \int_a^b \frac{\partial \rho}{\partial t} r \, dr \, z \, dz \, d\theta \tag{2.26}$$

$$= -C_{pK} + \mathcal{J}_p^{(\Phi)} - D_p \tag{2.27}$$

where C_{pK} is defined by equation (2.22). The flux of potential energy through the boundaries is

$$\mathcal{J}_p^{(\Phi)} = \frac{-g}{\rho_0} \int_0^{\Phi} \int_0^d [r u \rho]_{r=a}^{r=b} z \, dz \, d\theta - \frac{g}{\rho_0} \int_0^{\Phi} \int_a^b [w \rho z]_{z=0}^{z=d} r \, dr \, d\theta \tag{2.28}$$

and the rate of dissipation of potential energy is given by

$$D_p = -g \int_0^d \int_0^{\Phi} \int_a^b \left\{ \rho_1 + 2\rho_2 (T - T_0) \right\} \tilde{\Delta}^2(\kappa, T) \, r \, dr \, z \, dz \, d\theta. \tag{2.29}$$

3 THE GRID SCHEME AND INITIALIZATION OF DATA

In a meridional plane, the radial and vertical coordinates retain their dispositions in the axisymmetric scheme. The azimuthal grid is arranged as shown in Figure 1: the zonal velocity gridpoints are intermediate in azimuth between the gridpoints for the remaining fields. Since the azimuthal grids are

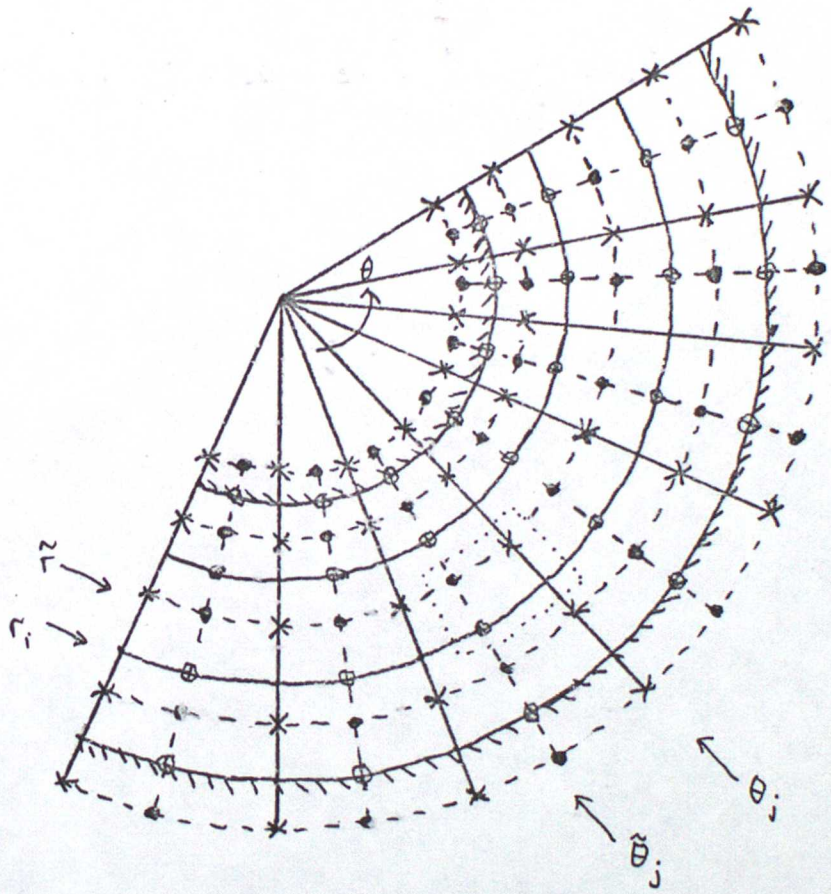

Figure 1

The finite-difference grid seen from above. The location of the boundaries is marked by shading; the (i,j) cell is delineated by the dotted line. Flow variables are evaluated at the following points:

- ⊙ - vertical velocity w, pressure π, and temperature T
- ⊕ - radial velocity u
- ✕ - azimuthal velocity v

regularly spaced there is only one grid-increment $(\delta\theta)$. Both the principal (θ_j) and intermediate $(\tilde{\theta}_j)$ grids are defined for all values of j, $1 \leq j \leq J$. The points are arranged so that the angle $\theta_J + \delta\theta$ is equivalent to θ_1, to establish periodicity.

The program is written to allow initialisation of the data in two ways. (a) The gridpoints and fields of a steady axisymmetric integration are used, the fields being replicated at each azimuthal gridpoint. (b) The radial and vertical gridpoints are determined ab initio from the given parameters, in an analogous fashion to that used in the axisymmetric scheme. In either case further data is required to define the periodic angle (Φ) and the number of azimuthal gridpoints (J).

To permit waves to develop in an initially axisymmetric state it is necessary to apply a perturbation. Two forms of perturbation have been programmed, both applied to the temperature field. (a) A wave (of any wavenumber and amplitude) is superimposed in the region $\theta = 0 - \Phi$, $r = \frac{a+b}{2}$, $z = \frac{d}{2}$. (b) A point perturbation of specified magnitude is applied at $\theta = 0$, $r = \frac{a+b}{2}$, $z = \frac{d}{2}$. Since in general $r = \frac{a+b}{2}, z = \frac{d}{2}$ is not the position of a gridpoint for T, in practice the perturbation is applied to the four gridpoints in each meridional plane which surround this position. Note that although the prediction equations conserve symmetry even in finite difference form (provided that the gridpoints themselves possess the appropriate symmetry), the use of Fourier transforms in the determination of the pressure field introduces round off-errors which vary with azimuth, thus allowing waves to develop eventually even in the absence of a primary perturbation.

Since there are now additional restrictions (of the type described in FP § 4.2) on the timestep, because of the possibility of variation in the azimuthal direction, the program also allows the timestep (Δt) to be changed from that used in the preceding axisymmetric integration. Accompanying this, the axisymmetric fields at times $t = m\Delta t$ and $t = (m-1)\Delta t$ are smoothed in the usual fashion described in FP § 5.2.

4. THE FINITE-DIFFERENCE EQUATIONS

4.1 Finite-difference form of the governing equations

Using the schemes described in FP § 4, the governing equations (2.1)-(2.3) are transformed into finite-difference prediction equations for u, v, w, T as follows:

$$\delta_t \bar{u}^t = -a_1(u) - \delta_r \pi + [2\Omega v]^* + \left[\frac{v^2}{r}\right]^* + D_u^\dagger \tag{4.1}$$

$$\delta_t \bar{v}^t = -\frac{1}{r} a_1(rv) - \frac{1}{r}\delta_\theta \pi - \frac{2\Omega}{r}\overline{(ru)}^r + D_v^\dagger \tag{4.2}$$

$$\delta_t \bar{w}^t = -a_1(w) - \delta_z \pi + [\sigma]^* + D_w^\dagger \tag{4.3}$$

$$\delta_t \bar{T}^t = -a_2(T) + D_T^\dagger \tag{4.4}$$

As with the axisymmetric equations, the right-hand sides of the equations are evaluated at the points appropriate to the quantities on the left-hand side. The asterisk again denotes non-standard averaging, and the superscript $(\dagger)$ signifies that the Dufort-Frankel scheme is to be incorporated.

The buoyancy term is essentially unchanged:

$$[\sigma]^{*m}_{i,j,k} = -g\left\{\rho_1\left(\overline{T}^{z\,m}_{i,j,k} - T_0\right) + \rho_2\left(T^m_{i,j,k} - T_0\right)\left(T^m_{i,j,k+1} - T_0\right)\right\} \tag{4.5}$$

The Coriolis and centrifugal term in the meridional velocity are written as follows:

$$[2\Omega v]^* = \frac{\Omega}{\delta\tilde{r}_{i+1}}\left(\bar{V}^\theta_{i,j,k}\,\delta r_i + \bar{V}^\theta_{i+1,j,k}\,\delta r_{i+1}\right) \tag{4.6}$$

$$\left[\frac{v^2}{r}\right]^* = \frac{V_{i,j,k}V_{i+1,j,k} + V_{i,j-1,k}V_{i+1,j-1,k}}{4\,\delta\tilde{r}_{i+1}}\left(\frac{r_{i+1}}{r_i} - \frac{\tilde{r}_i}{r_{i+1}}\right) \tag{4.7}$$

The adaptations of the advective terms may be written

$$a_2(f) = \frac{1}{r}\delta_r\left[(\widetilde{ru})\overline{f}^{\,r}\right] + \delta_z\left[\widetilde{w}\,\overline{f}^{\,z}\right] + \frac{1}{r}\delta_\theta\left[\widetilde{v}\,\overline{f}^{\,\theta}\right] \qquad (4.8)$$

and

$$a_1(f) = \frac{1}{2}\left\{a_2(f) + \frac{1}{r}\overline{\left[(r\widetilde{u})\,\delta_r f\right]}^{\,r} + \overline{\left[\widetilde{w}\,\delta_z f\right]}^{\,z} + \frac{1}{r}\overline{\left[\widetilde{v}\,\delta_\theta f\right]}^{\,\theta}\right\} \qquad (4.9)$$

when the tilde again means that the indicated quantity may need to be interpolated on to the appropriate mesh point.

The diffusion terms are

$$D_u = \frac{1}{r^2}\delta_r\left\{\nu r^3\,\delta_r\left(\frac{u}{r}\right)\right\} + \delta_z\left\{\overline{\overline{\nu}}^{\,r,z}\,\delta_z u\right\} + \frac{1}{r^2}\delta_\theta\left\{\overline{\overline{\nu}}^{\,r,\theta}\left(\delta_\theta u - \overline{v}^{\,r}\right)\right\} \ldots . \qquad (4.10)$$

$$\ldots\ldots + \frac{1}{r}\delta_r r\,\delta_r\overline{(ru)}^{\,r} + \frac{1}{r^2}\delta_r\overline{(rv)}^{\,\theta}\delta_\theta\overline{\nu}^{\,r} + \delta_r\overline{w}^{\,z}\,\delta_z\overline{\nu}^{\,r}$$

$$D_v = \frac{1}{r^2}\delta_r\left\{\overline{\overline{\nu}}^{\,r,\theta}\,r^3\,\delta_r\left(\frac{v}{r}\right)\right\} + \delta_z\left\{\overline{\overline{\nu}}^{\,\theta,z}\,\delta_z v\right\} \ldots\ldots$$

$$\ldots\ldots + \frac{1}{r^2}\delta_\theta\left\{r\left(\overline{u}^{\,r} + \delta_\theta v\right)\right\} + \frac{1}{r}\delta_r\overline{r}^{\,\theta}\,\delta_\theta\overline{u}^{\,r} \ldots\ldots \qquad (4.11)$$

$$\ldots + \frac{1}{r^2}\delta_\theta\overline{\nu}^{\,\theta}\,\delta_\theta v + \frac{1}{r}\delta_z\overline{\nu}^{\,z}\,\delta_\theta\overline{w}^{\,z}$$

$$D_\omega = \frac{1}{r}\delta_r\left\{\bar{\bar{\nu}}^{r,z} r \delta_r \omega\right\} + \delta_z\left\{\nu \delta_z \omega\right\} + \frac{1}{r^2}\delta_\theta\left\{\bar{\bar{\nu}}^{\theta,z}\delta_\theta \omega\right\} \ldots$$

(4.12)

$$\ldots + \delta_z \bar{u}^r \delta_r \bar{\nu}^z + \frac{1}{r}\delta_z \bar{v}^\theta \delta_\theta \bar{\nu}^z + \delta_z \bar{w}^z \delta_z \nu$$

$$D_T = \frac{1}{r}\delta_r\left\{\bar{k}^r r \delta_r T\right\} + \delta_z\left\{\bar{k}^z \delta_z T\right\} \ldots$$

(4.13)

$$\ldots + \frac{1}{r^2}\delta_\theta\left\{\bar{k}^\theta \delta_\theta T\right\}$$

The viscosity and conductivity are functions of temperature as in the axisymmetric scheme (equation FP 4.19). The continuity equation is now of the form

$$\Delta = \frac{1}{r}\delta_r(ru) + \delta_z w + \frac{1}{r}\delta_\theta v = 0 \qquad (4.14)$$

4.2 Integrals of the finite difference equations

(a) Angular momentum

The finite-difference angular momentum integral is $n\rho_0 A$ where

$$A^m = \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \tilde{r}_i^2 \, v^m_{i,j,k} \, \delta r_i \, \delta z_k \, \delta\theta \tag{4.15}$$

so that integration of equation (4.2) yields

$$\delta_t \bar{A}^t = \mathcal{J}_A^{(D)} + \mathcal{J}_A^{(V)} + \mathcal{J}_A^{(R)} + \delta_A + \delta_B \tag{4.16}$$

The dynamic flux of angular momentum through the boundaries is

$$\mathcal{J}_A^{(D)\,m} = \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \tilde{r}_i^2 \, \delta r_i \left\{ \bar{\omega}^{\theta\,m}_{i,j+\frac{1}{2},1} \, \bar{v}^{z\,m}_{i,j,\frac{3}{2}} - \bar{\omega}^{\theta}_{i,j+\frac{1}{2},K-1} \, \bar{v}^{z\,m}_{i,j,K-\frac{1}{2}} \right\} \cdots$$

$$\cdots + \delta\theta \sum_{j=1}^{J} \sum_{k=2}^{K-1} \delta z_k \left\{ r_1 \, \bar{u}^{\theta\,m}_{1,j+\frac{1}{2},k} \, \overline{(\tilde{r}v)}^{r\,m}_{\frac{3}{2}jk} - r_{I-1} \, \bar{u}^{\theta\,m}_{I-1,j+\frac{1}{2},k} \, \overline{(\tilde{r}v)}^{r\,m}_{I-\frac{1}{2}jk} \right\}. \tag{4.17}$$

Note that for $j = J$, $f(j+\frac{1}{2}) = \dfrac{f(J) + f(1)}{2}$.

The viscous contribution is

$$\mathcal{J}_A^{(V)\,m} = \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \tilde{r}_i^2 \, \delta r_i \left\{ \bar{\bar{\nu}}^{\theta,z\,m}_{i,j+\frac{1}{2},K-\frac{1}{2}} \left(\delta_z v\right)^m_{ijK-\frac{1}{2}} - \bar{\bar{\nu}}^{\theta,z\,m}_{i,j+\frac{1}{2},\frac{3}{2}} \left(\delta_z v\right)^m_{ij\frac{3}{2}} \right\} \cdots$$

$$\cdots + \delta\theta \sum_{j=1}^{J} \sum_{k=2}^{K-1} \delta z_k \left\{ \bar{\bar{\nu}}^{\theta,r\,m}_{I-\frac{1}{2},j+\frac{1}{2},k} \, r_{I-1}^3 \left[\delta_r\left(\frac{v}{r}\right)\right]^m_{I-\frac{1}{2},j,k} \cdots \right.$$

$$\left. \cdots - \bar{\bar{\nu}}^{\theta,r}_{\frac{3}{2},j+\frac{1}{2},k} \, r_1^3 \left[\delta_r\left(\frac{v}{r}\right)\right]^m_{\frac{3}{2}jk} \right\} + \delta\theta \sum_{j=1}^{J} \sum_{k=2}^{K-1} \sum_{i=2}^{I-1} \delta r_i \, \delta z_k \times \cdots \tag{4.18}$$

$$\cdots \left\{ \tilde{r}_i \left[\delta_r v\right]^m_{i,j+\frac{1}{2},k} \left[\delta_\theta u\right]^m_{i-\frac{1}{2},j+\frac{1}{2},k} + \left[\delta_\theta r\right]^m_{ijk} \left[\delta_\theta v\right]^m_{i,j+\frac{1}{2},k} \cdots \right.$$

$$\left. \cdots + \tilde{r}_i \left[\delta_z v\right]^m_{i,j+\frac{1}{2},k} \left[\delta_\theta w\right]^m_{i,j+\frac{1}{2},k-\frac{1}{2}} \right\}.$$

The mass flux term is again

$$\mathcal{J}_A^{(R)\,m} = -2\Omega \sum_{i=2}^{I-1} \mathcal{M}_i^m \tilde{r}_i \delta r_i \qquad (4.19)$$

where the meridional mass flux is

$$\mathcal{M}_i^m = \delta\theta \sum_{k=2}^{K-1} \sum_{j=1}^{J} \delta z_k \overline{(ru)}^{r\,m}_{i-\frac{1}{2},j,k} \,. \qquad (4.20)$$

The error introduced by use of the Dufort-Frankel scheme is

$$\delta_A^m = -\Delta t^2 \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \tilde{r}_i^2 \left[\delta_t(\delta_t v)\right]^m_{ijk} \cdots\cdots$$

$$\cdots\cdots \times \left\{ \frac{\delta z_k}{r_i^3} \left[\frac{\overline{\bar{v}^r r^3}^{r\,m}}{\delta r} \right]_{ijk} + \delta r_i \left[\frac{\overline{\bar{v}^z}^{z\,m}}{\delta z} \right]_{ijk} + \frac{\bar{v}^{\theta\,m}_{ijk}}{\tilde{r}_i^2 \delta\theta^2} \delta r_i \delta z_k \right\} . \qquad (4.21)$$

The divergence error is

$$\delta_B^m = \frac{\delta\theta}{2} \sum_{j=1}^{J} \sum_{k=2}^{K-1} \sum_{i=2}^{I-1} \tilde{r}_i^2 V^m_{ijk} \Delta^m_{ijk} \delta r_i \delta z_k \,. \qquad (4.22)$$

As in the axisymmetric scheme both errors vanish in a steady state.

(b) Heat Balance

The total heat content of the fluid is $n c \rho_0 Q$, where

$$Q = \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \tilde{r}_i T^n_{ijk} \delta r_i \delta z_k \qquad (4.23)$$

and its evolution is described by the integral of equation (4.4), namely

$$\delta_t \bar{Q}^{t} = \mathcal{J}_a^{(D)} + \mathcal{J}_a^{(V)} + \delta_Q \qquad (4.24)$$

where the dynamic heat flux through the boundaries is

$$\mathcal{J}_a^{(D)\,m} = -\delta\theta \sum_{j=1}^{J} \left[\sum_{i=2}^{I-1} \left\{ \omega^m_{ijK-1}\, \bar{T}^{z\,m}_{ijK-\frac{1}{2}} - \omega^m_{ij,1}\, \bar{T}^{z\,m}_{ij,\frac{3}{2}} \right\} \tilde{r}_i\, \delta r_i \ldots\ldots \right.$$

$$\left. \ldots\ldots + \sum_{k=2}^{K-1} \left\{ r_{I-1}\, u^m_{I-1,jk}\, \bar{T}^{r\,m}_{I-\frac{1}{2}jk} - r_1\, u^m_{1jk}\, \bar{T}^{r\,m}_{\frac{3}{2},j,k} \right\} \delta z_k \right] \qquad (4.25)$$

and the conductive flux through the walls is

$$\mathcal{J}_a^{(V)\,m} = \delta\theta \sum_{j=1}^{J} \left[\sum_{i=2}^{I-1} \left\{ \bar{K}^{z\,m}_{ijK-\frac{1}{2}} (\delta_z T)^m_{ijK-\frac{1}{2}} - \bar{K}^{z\,m}_{ij,\frac{3}{2}} (\delta_z T)^m_{ij\frac{3}{2}} \right\} \tilde{r}_i\, \delta r_i \ldots\ldots \right.$$

$$\left. \ldots\ldots + \sum_{k=2}^{K-1} \left\{ \bar{K}^{r\,m}_{I-\frac{1}{2}jk} (\delta_r T)^m_{I-\frac{1}{2}jk} - \bar{K}^{r\,m}_{\frac{3}{2}jk} (\delta_r T)^m_{\frac{3}{2}jk} \right\} \delta z_k \right. \qquad (4.26)$$

The Dufort-Frankel error is

$$\delta^m_Q = -\Delta t^2 \delta\theta \sum_{1}^{J} \sum_{2}^{K-1} \sum_{2}^{I-1} \left[\delta_t (\delta_t T) \right]^m_{ijk} \left\{ \delta z_k \left(\overline{\frac{r\, \bar{K}^r}{\delta r}} \right)^{r\,m}_{ijk} \ldots\ldots \right.$$

$$\left. \ldots\ldots + \tilde{r}_i \delta r_i \left(\overline{\frac{\bar{K}^z}{\delta z}} \right)^{z\,m}_{ijk} + \frac{\overline{[\bar{K}^\theta]}^{\theta}}{\tilde{r}\, \delta\theta^2} \delta r_i \delta z_k \right\} \qquad (4.27)$$

(c) Kinetic Energy

The total kinetic energy of the fluid, defining ξ_K (again at time $t = (n+\frac{1}{2})\Delta t$ to be

$$\mathcal{E}_K^{m+\frac{1}{2}} = \frac{1}{2} \sum\sum\sum \left\{ u^m u^{m+1} + v^m v^{m+1} + w^m w^{m+1} \right\} r\, \delta r\, \delta\theta\, \delta z \qquad (4.28)$$

is now $\rho_0 \mathcal{E}_K$. The rate of change of $\mathcal{E}_K$ in the restricted case where any flow normal to the walls is constant, is

$$\left(\delta_t \mathcal{E}_K\right)^m = \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-2} \sum_{k=2}^{K-1} u^m_{ijk} \left(\delta_t \bar{u}^t\right)^m_{ijk} r_i\, \delta\tilde{r}_{i+1}\, \delta z_k \ \ldots\ldots$$

$$\ldots + \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} v^m_{ijk} \left(\delta_t \bar{v}^t\right)^m_{ijk} \tilde{r}_i\, \delta r_i\, \delta z_k \ \ldots\ldots \qquad (4.29)$$

$$\ldots\ldots + \delta\theta \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-2} w^m_{ijk} \left(\delta_t \bar{w}^t\right)^m_{ijk} \tilde{r}_i\, \delta r_i\, \delta\tilde{z}_{k+1}$$

which yields, after substitution using equations (4.1)-(4.3),

$$\delta_t \mathcal{E}_K = C_{PK} + \mathcal{J}_K^{(D)} - D_K + \delta_K^{(R)} + \delta_K^{(D)} \qquad (4.30)$$

with terms defined as follows. The flux of kinetic energy and internal energy into the fluid is

$$\mathcal{J}_K^{(D)\,m} = -\frac{1}{2} \sum_{j=1}^{J} \delta\theta \sum_{k=2}^{K-1} \delta z_k \left\{ u^m_{I-1,j,k}\, u^m_{I-2\,jk} \left(\overline{ru}\right)^{r\,m}_{I-\frac{3}{2}\,jk} - u^m_{1jk}\, u^m_{2jk} \left(\overline{ru}\right)^{r\,m}_{\frac{3}{2}\,jk} \right\} \ldots$$

$$\ldots - \frac{1}{2} \sum_{j=1}^{J} \delta\theta \sum_{i=2}^{I-2} r_i\, \delta\tilde{r}_{i+1} \left\{ u^m_{ijk}\, u^m_{ij\,k-1}\, \bar{w}^{r\,m}_{i+\frac{1}{2}\,j\,k-1} - u^m_{ij1}\, u^m_{ij2}\, \bar{w}^{r\,m}_{i+\frac{1}{2}\,j1} \right\} \ldots\ldots$$

$$\ldots - \frac{1}{2} \sum_{j=1}^{J} \delta\theta \sum_{k=2}^{K-2} \delta\tilde{z}_{k+1} \left\{ w^m_{I-1\,jk}\, w^m_{Ijk} \left(\overline{ru}\right)^{z\,m}_{Ij\,k+\frac{1}{2}} - w^m_{2jk}\, w^m_{1jk} \left(\overline{ru}\right)^{z\,m}_{1j\,k+\frac{1}{2}} \right\} \ldots \qquad (4.31)$$

$$\ldots -\frac{1}{2}\sum_{j=1}^{J}\delta\theta\sum_{i=2}^{I-1}\tilde{r}_i\delta r_i\left\{w^m_{ijk-2}\,w^m_{ijk-1}\,\bar{\bar{w}}^{zm}_{ijk-3/2}-w^m_{ij1}\,w^m_{ij2}\,\bar{\bar{w}}^{zm}_{ij3/2}\right\}\ldots\ldots$$

$$\ldots -\frac{1}{2}\sum_{j=1}^{J}\delta\theta\sum_{k=2}^{K-1}\delta z_k\left\{\overline{(ru)}^{\theta m}_{I-1\,j+1/2\,k}(rv)^m_{I\,jk}\left(\frac{v}{r}\right)^m_{I-1\,jk}-\overline{(ru)}^{\theta m}_{1j+1/2\,k}(rv)^m_{1jk}\left(\frac{v}{r}\right)^m_{2jk}\right\}\ldots$$

$$\ldots -\frac{1}{2}\sum_{j=1}^{J}\delta\theta\sum_{i=2}^{I-1}\tilde{r}_i\delta r_i\left\{v^m_{ijk-1}\,v^m_{ijk}\,\bar{\bar{w}}^{\theta m}_{ij+1/2\,k-1}-v^m_{ij1}\,v^m_{ij2}\,\bar{\bar{w}}^{\theta m}_{ij+1/2\,1}\right\}\ldots\ldots$$

$$\ldots -\sum_{j=1}^{J}\delta\theta\left\{\sum_{k=2}^{K-1}\delta z_k\left(\Pi^m_{I-1jk}(ur)^m_{I-1jk}-\Pi^m_{2jk}(ur)^m_{1jk}\right)+\sum_{i=2}^{I-1}\tilde{r}_i\delta r_i\left(\Pi^m_{ijk-1}w^m_{ijk-1}-\Pi^m_{ij2}w^m_{ij1}\right)\right\}$$

Note that the continuity equation has been used to integrate the pressure-work term, thus giving rise to the error term

$$\delta^{(D)\,m}_k=\sum_{j=1}^{J}\sum_{i=2}^{I-1}\sum_{k=2}^{K-1}\tilde{r}_i\,\Pi^m_{ijk}\,\Delta^m_{ijk}\,\delta r_i\,\delta z_k\,\delta\theta \qquad (4.32)$$

The rate of conversion of potential to kinetic energy is

$$C^m_{pK}=\sum_{j=1}^{J}\sum_{i=2}^{I-1}\sum_{k=2}^{K-1}w^m_{ijk}\left[\sigma\right]^{*m}_{ijk}\tilde{r}_i\,\delta r_i\,\delta\tilde{z}_{k+1}\,\delta\theta \qquad (4.33)$$

The rate of dissipation of kinetic energy is

$$D^m_k=-\sum_{j=1}^{J}\delta\theta\sum_{i=2}^{I-2}\sum_{k=2}^{K-1}r_i\,u^m_{ijk}\left(D^r_u\right)^m_{ijk}\delta\tilde{r}_{i+1}\,\delta z_k\ldots$$

$$\ldots-\sum_{j=1}^{J}\delta\theta\sum_{i=2}^{I-1}\sum_{k=2}^{K-1}\tilde{r}_i\,v^m_{ijk}\left(D^t_v\right)^m_{ijk}\delta r_i\,\delta z_k\ldots\ldots \qquad (4.34)$$

$$\ldots-\sum_{j=1}^{J}\delta\theta\sum_{i=2}^{I-1}\sum_{k=2}^{K-2}\tilde{r}_i\,w_{ijk}\left(D^t_w\right)^m_{ijk}\delta r_i\,\delta\tilde{z}_{k+1}$$

The Coriolis term error

$$\delta_K^{(R)\,m} = \Omega \delta\theta \sum_{j=1}^{J} \sum_{k=2}^{K-1} \delta_{\tilde{z}k} \left\{ -\overline{(ru)}^{\theta\,m}_{i\,j+\frac{1}{2}\,k} \left(\delta_r v\right)^m_{2jk} \cdots \overline{(ru)}^{\theta\,m}_{I-1\,j+\frac{1}{2}\,k} \left(\delta_r v\right)^m_{I-1\,jk} \right\} \quad (4.35)$$

is again zero if there is no source of fluid through the sidewalls.

(d) Potential Energy

The potential energy of the system is defined as $\eta \rho_o \mathcal{E}_p$, when

$$\mathcal{E}_p = - \sum_{j=1}^{J} \delta\theta \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \sigma^m_{ijk} \left(\tilde{z}_k - \frac{d}{2}\right) \tilde{r}_i \, \delta r_i \, \delta z_k \quad (4.36)$$

so that a homogeneous fluid has zero potential energy. Thus

$$\delta_t \overline{\mathcal{E}}^t_p = - \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \sum_{j=1}^{J} \delta\theta \, \tilde{r}_i \tilde{z}_k \, \delta r_i \, \delta z_k \left(\delta_t \overline{\sigma}\right)^n_{ijk} \quad (4.37)$$

where from equations (2.7), (4.4) and (4.5) we get

$$\delta_t \overline{\sigma}^t = -g \left\{ \rho_1 + 2\rho_2 \left(\overline{\overline{T}}^{tt} - T_o \right) \right\} \left\{ D_T^t - Q_2(T) \right\} \quad (4.38)$$

so that

$$\delta_t \overline{\mathcal{E}}^t_p = \mathcal{J}_p^{(\sigma)} - C_{pK} - D_p + \delta_r \quad (4.39)$$

where C_{pK} is the energy conversion term (4.33). The flux of potential energy through the boundaries is

$$\mathcal{J}_p^{(\sigma)\,m} = \sum_{i=2}^{I-1} \sum_{j=1}^{J} \tilde{r}_i \delta r_i \delta\theta \left\{ \tilde{z}_{K-1} w^m_{ij\,k-1} \left[\sigma\right]^{r\,m}_{ijK-1} - \tilde{z}_2 \, w^m_{ij\,1} \left[\sigma\right]^{*\,m}_{ij\,1} \right\} \cdots$$

$$\cdots + \sum_{k=2}^{K-1} \sum_{j=1}^{J} \tilde{z}_k \delta z_k \delta\theta \left\{ r_{I-1} u^m_{I-1jk} \overline{\left[\sigma\right]}^{r\,*\,m}_{I-\frac{1}{2}jk} - r_1 u^m_{1jk} \overline{\left[\sigma\right]}^{r\,*\,m}_{\frac{3}{2}jk} \right\} \quad (4.40)$$

and the rate of dissipation of potential energy via thermal diffusion is expressed by the term

$$\mathcal{D}_p = g \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \sum_{j=1}^{J} \left\{ \rho_1 + 2\rho_2 \left(\overline{\overline{T}}^{tt} - T_0 \right)^m_{ijk} \right\} \left(\mathcal{D}_T^t \right)^m_{ijk} \tilde{z}_k \tilde{r}_i \delta r_i \delta z_k \delta\theta \qquad (4.41)$$

The error due to the quadratic nature of the equation of state is

$$\delta_p = -g\rho_2 \sum_{j=1}^{J} \sum_{i=2}^{I-1} \sum_{k=2}^{K-1} \left\{ \left(T^m_{ijk} - T_0 \right)^2 \Delta^m_{ijk} + 2\Delta t^2 \left[\delta_t \left(\delta_t T \right) \right]^m_{ijk} \right\} \cdots$$

$$\cdots \times \left[\alpha_2(T) \right]^m_{ijk} \tilde{r}_i \tilde{z}_k \delta r_i \delta z_k \delta\theta \qquad (4.42)$$

(e) Asymptotic form of the integrals in a steady state with no source of fluid at the sidewalls.

These forms are the same as in the axisymmetric case (FP 4.50).

$$\delta_t \bar{A}^t = \mathcal{J}_A^{(v)}$$

$$\delta_t \bar{Q}^t = \mathcal{J}_Q^{(v)}$$

$$\delta_t \mathcal{E}_k = C_{PK} - D_K \qquad (4.43)$$

and $$\delta_t \bar{\mathcal{E}}_p^t = -C_{PK} - D_P$$

4.3 The Poisson equation

The Poisson equation for pressure is complicated by the third term in the continuity equation (4.14). Writing the equations of motion (4.1)-(4.3) as

$$\delta_t \bar{u}^t = -\delta_r \Pi + S_u - \tfrac{1}{2} \Delta t^2 \varepsilon_u \delta_t \left(\delta_t u \right)$$

$$\delta_t \bar{v}^t = -\frac{1}{r} \delta_\theta \Pi + S_v - \tfrac{1}{2} \Delta t^2 \varepsilon_v \delta_t \left(\delta_t v \right) \qquad (4.44)$$

$$\delta_t \bar{w}^t = -\delta_z \Pi + S_w - \tfrac{1}{2} \Delta t^2 \varepsilon_w \delta_t \left(\delta_t w \right)$$

(compare FP 4.51), where

$$(\varepsilon_u)^m_{ijk} = \frac{1}{r_i^2 \delta\tilde{r}_{i+1}} \left\{ \frac{\nu^m_{i+1jk}\, \tilde{r}^3_{i+1}}{\delta r_{i+1}} + \frac{\nu^m_{ijk}\, \tilde{r}^3_i}{\delta r_i} \right\} + \frac{1}{\delta z_k} \left\{ \frac{\overline{\overline{\tau}}^{rz\ m}_{i+\frac{1}{2}\,j\,k+\frac{1}{2}}}{\delta \tilde{z}_{k+1}} + \frac{\overline{\overline{\tau}}^{rz\ m}_{i+\frac{1}{2}\,j\,k-\frac{1}{2}}}{\delta \tilde{z}_k} \right\} \ldots .$$

$$\ldots . + \frac{1}{r_i^2 \delta\theta^2} \left\{ \overline{\overline{\tau}}^{r\theta\ m}_{i+\frac{1}{2}\,j+\frac{1}{2}\,k} + \overline{\overline{\tau}}^{r\theta\ m}_{i+\frac{1}{2}\,j-\frac{1}{2}\,k} \right\}$$

$$(\varepsilon_v)^m_{ijk} = \frac{1}{\tilde{r}_i^3 \delta r_i} \left\{ \frac{\overline{\overline{\tau}}^{r\theta\ m}_{i+\frac{1}{2}\,j+\frac{1}{2}\,k}\, r_i^3}{\delta \tilde{r}_{i+1}} + \frac{\overline{\overline{\tau}}^{r\theta\ m}_{i-\frac{1}{2}\,j+\frac{1}{2}\,k}\, r_{i-1}^3}{\delta r_i} \right\} + \frac{1}{\delta z_k} \left\{ \frac{\overline{\overline{\tau}}^{\theta z\ m}_{i\,j+\frac{1}{2}\,k+\frac{1}{2}}}{\delta \tilde{z}_{k+1}} + \frac{\overline{\overline{\tau}}^{\theta z}_{i\,j+\frac{1}{2}\,k-\frac{1}{2}}}{\delta \tilde{z}_k} \right\} \ldots .$$

$$\ldots . + \frac{1}{\tilde{r}_i^2 \delta\theta^2} \left\{ \nu^m_{i\,j+1\,k} + \nu^m_{ijk} \right\} \qquad (4.45)$$

and $$(\varepsilon_w)^m_{ijk} = \frac{1}{\tilde{r}_i \delta r_i} \left\{ \frac{r\,\overline{\overline{\tau}}^{rz\ m}_{i+\frac{1}{2}\,jk+\frac{1}{2}}}{\delta \tilde{r}_{i+1}} + \frac{r_{i-1}\,\overline{\overline{\tau}}^{rz\ m}_{i-\frac{1}{2}\,jk+\frac{1}{2}}}{\delta \tilde{r}_i} \right\} + \frac{1}{\delta \tilde{z}_k} \left\{ \frac{\nu^m_{ijk+1}}{\delta z_{k+1}} + \frac{\nu^m_{ijk}}{\delta z_k} \right\} \ldots .$$

$$\ldots . + \frac{1}{\tilde{r}_i^2 \delta\theta^2} \left\{ \overline{\overline{\tau}}^{\theta z\ m}_{i\,j+\frac{1}{2}\,k+\frac{1}{2}} + \overline{\overline{\tau}}^{\theta z\ m}_{i\,j-\frac{1}{2}\,k+\frac{1}{2}} \right\}$$

and redefining $\underset{\sim}{u} = (u, v, w)$, $\underset{\sim}{S} = (S_u, S_v, S_w)$ and the operators $\underset{\sim}{\nabla}\cdot$ and ∇^2 as

$$\underset{\sim}{\nabla}\cdot(a, b, c) = \frac{1}{r}\delta_r(ra) + \delta_z c + \frac{1}{r}\delta_\theta b \qquad (4.46)$$

and

$$\nabla^2 c = \frac{1}{r}\delta_r(r\delta_r c) + \delta_z(\delta_z c) + \frac{1}{r^2}\delta_\theta(\delta_\theta c) \qquad (4.47)$$

equations (4.44) reduce to

$$\delta_t \bar{\Delta}^t \equiv \delta_t \overline{\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right)}^t = -\nabla^2 \Pi + \underset{\sim}{\nabla}\cdot\underset{\sim}{S} - \ldots .$$

$$\ldots . \frac{\Delta t^2}{2} \underset{\sim}{\nabla}\cdot\left\{\varepsilon_u \delta_t(\delta_t u), \varepsilon_v \delta_t(\delta_t v), \varepsilon_w \delta_t(\delta_t w)\right\} \qquad (4.48)$$

so that continuity is satisfied, after neglecting the final term as in the axisymmetric case (FP 4.56), by solving the equation

$$\Delta^{m+1} = -\nabla^2 \Pi^m + \underset{\sim}{\nabla}\cdot\underset{\sim}{S}^m + \frac{1}{2\Delta t}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right)^{m-1} = 0 \qquad (4.49)$$

The boundary conditions are those given in FP 4.57, viz

$$(\delta_r \Pi)^m_{ijk} = (S_u)^m_{ijk} - (\delta_t \bar{u}^t)^m_{ijk}$$

$$(\delta_r \Pi)^m_{I-1jk} = (S_u)^m_{I-1jk} - (\delta_t \bar{u}^t)^m_{I-1jk}$$

$$(\delta_z \Pi)^m_{ij1} = (S_w)^m_{ij1} - (\delta_t \bar{w}^t)^m_{ij1} \qquad (4.50)$$

and $$(\delta_z \Pi)^m_{ijK-1} = (S_w)^m_{ijK-1} - (\delta_t \bar{w}^t)^m_{ijK-1}$$

plus the requirement of periodicity

$$\Pi_{iJ+1k} = \Pi_{i1k}$$

$$\Pi_{iJk} = \Pi_{i0k} \qquad (4.51)$$

The method of solution of the Poisson equation is discussed in § 5.1.

5 METHOD OF SOLUTION

5.1 The Poisson Equation

The method of solution of the Poisson equation is adapted from that of Williams (1969). The equation may be written

$$\nabla^2 \Pi = F \qquad (5.1)$$

where

$$F^m = \underset{\sim}{\nabla} \cdot \underset{\sim}{S}^m + \frac{1}{2\Delta t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u})^{m-1} \qquad (5.2)$$

To ensure periodicity both F and Π are expanded as a Fourier Series:

$$\Pi(r,\theta,z) = \sum_{\ell=1}^{J} P_\ell(r,z)\, H_\ell(\theta)$$

$$F(r,\theta,z) = \sum_{\ell=1}^{J} Q_\ell(r,z)\, H_\ell(\theta) \qquad (5.3)$$

where H_ℓ are the usual trigonometric functions. Substituting (5.3) into (5.1) yields

$$\frac{1}{r}\delta_r\left[r\delta_r \sum_\ell P_\ell H_\ell\right] + \frac{1}{r^2}\delta_\theta\left[\delta_\theta \sum_\ell P_\ell H_\ell\right] + \delta_z\left[\delta_z \sum_\ell P_\ell H_\ell\right] = \sum_\ell Q_\ell H_\ell \qquad (5.4)$$

Since H_ℓ are orthogonal and functions of θ only, this gives a separate equation for each eigenfunction:

$$Q_\ell = \frac{1}{r}\delta_r\left[r\delta_r P_\ell\right] + \frac{P_\ell}{r^2}\left[\frac{\delta_\theta\{\delta_\theta H_\ell\}}{H_\ell}\right] + \delta_z\left[\delta_z P_\ell\right]$$

$$= \frac{\lambda_\ell P_\ell}{r^2} + \frac{1}{r}\delta_r\left[r\delta_r P_\ell\right] + \delta_z\left[\delta_z P_\ell\right] \qquad (5.5)$$

where $$\lambda_\ell = \frac{1}{H_\ell}\delta_\theta(\delta_\theta H_\ell) \qquad (5.6)$$

The values of λ_ℓ are found to be

$$\lambda_\ell = \frac{2}{(\delta\theta)^2}\left(1 - \cos\frac{2\pi\ell}{J-1}\right) \qquad (5.7)$$

for both $\sin(\ell\theta)$ and $\cos(\ell\theta)$ functions. Equation (5.5) is similar to FP 5.1, and is thus solved by alternating direction iteration as described for the axisymmetric program (FP 5.1). The boundary conditions are obtained by Fourier analysing the velocity source functions at the walls. When the coefficients P_ℓ have been obtained it is a trivial task to synthesise the pressure field.

5.2 Outline of Procedure

The general outline given in FP § 5.2 for obtaining an initial guess at the pressure field, for stepping the velocity and temperature fields, and for the suppression of time-splitting, also apply to the three dimensional scheme. Since the fields are already in an axisymmetric equilibrium state in the usual method of set up, it is generally sufficient to use 8 iterations in the Poisson equation solution even at the start of the integration.

The program is inevitably complicated by the need to store large amounts of data, so large in fact that it is necessary to keep most of it out of main core. It is convenient to store meridional sections (ie constant j, though the corresponding value of θ is $\frac{\delta\theta}{2}$ greater for v than for the remaining fields) in sequence from $j=1$ to $j=J+2$, all fields at given j being kept together. The final two sections are duplicates of the first two and are provided to simplify the problem of periodicity. The program naturally divides into three main stages, as described before. Four back-up data-sets are required, numbered 1, 2, 3 and 5. The fields are initially placed on data-set 1.

(i) Input fields are read from data-set 1. Four sections are present in core at any time, three being used (to allow δ_θ terms to be calculated while the fourth is being read. (The program allows reading/writing to be performed simultaneously with the computation.)

For each of the velocity components two source function fields are evaluated, one containing terms involved in the Dufort-Frankel scheme (coefficients of u^{m+1}_{ijk} and u^{m-1}_{ijk}), since these are treated differently when the pressure is obtained, and one containing the remaining terms (incorporating the velocity at time $m\delta t$). These fields are written to data-set 3. The guess of the pressure field is also determined and written to data-set 5. The source

function for the pressure (F in equation 5.1) is evaluated and retained in core.

(ii) The pressure guess is retrieved from data-set 5, and the Poisson equation solved as described in § 5.1. The new pressure field is retained in main core, overwriting the pressure source function which is no longer required. Since a Fourier analysis of the pressure is necessarily available, it is convenient to print it as a diagnostic; the format may be varied according to the problem by inclusion of an appropriate subroutine. It is also possible to obtain an estimate of the Fourier analysis of the temperature by means of the hydrostatic approximation:

$$T = \frac{1}{g\rho_1} \delta_z \Pi \qquad (5.8)$$

(iii) The input fields are again obtained from data-set 1, while the velocity source functions are retrieved from data-set 3. The velocity and temperature fields are stepped and written together with the new pressure field to data-set 2. The boundary conditions are also applied at this time. Note that because data-sets 2 and 5 are never employed simultaneously it is possible to use the same backing store for them; the difference is simply in the formatting of the data.

At the end of this sequence the output fields are stored on data-set 2 in the same format as previously on data-set 1. Thus the roles of data-sets 1 and 2 are interchanged, and the process may be repeated. However the meridional slice originally at the beginning of the data-set has been transferred to the end, and the remaining sections advanced accordingly: the original position reappears after J timesteps. Thus an indicator of the current position is maintained to allow the proper phase to be determined.

A fourth section of the program contains routines to perform tasks which are not required in every step. These include the evaluation of diagnostics, maintaining a recent dump to allow restartability, and saving the fields for later examination; they require only the reading of data-set 1. Smoothing of the fields is also easily accomplished, with transfer of the fields to data-set 2.

6 SUMMARY

A model has been described which is designed for the integration of the Navier-Stokes equations for the three-dimensional motion of a Boussinesq liquid contained in a rotating annulus. The finite difference scheme exhibits second-order accuracy on a non-uniform mesh, and is formulated so as to generate the fewest possible errors in the physically important integrals of the motion. These errors vanish in a steady state for a system that has no sources or sinks of fluid at the boundaries; during the time-dependent stage of an integration, these errors are non-zero, but they are always small, and do not promote computational instability.

The model is an extended version of our program for the integration of axisymmetric motions (Farnell and Plumb, 1975), developed on the lines of that of G P Williams (1969).

REFERENCES

1 Farnell L, and Plumb R A 1975 "Numerical integration of flow in a rotating annulus I: axisymmetric model", Occasional Note Met O 21 75/3.

2 Williams G P 1969 "Numerical integration of the three-dimensional Navier-Stokes equations for incompressible flow", J Fluid Mech, 37, 727-750.